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The analysis of a finite element method for the three-species Lotka–Volterra competition-diffusion with Dirichlet boundary conditions

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Abstract

A Galerkin finite element method is developed for the two dimensional/three dimensional nonlinear time-dependent three-species Lotka–Volterra competition-diffusion equations on a bounded domain. The existence and uniqueness of the solution to the numerical formulation are proved. An error estimate for the numerical solution is obtained. Numerical computations are carried out to examine the expected orders of accuracy in the error estimates.

Keywords: Three-species Lotka–Volterra competition-diffusion equations; Finite element; Error estimate; Two-dimensional; Three dimensional

1. Introduction

In this paper, we assume that all competing species occur by diffusion. Let Ω be a bounded domain in Euclidean space $\mathbb{R}^d$ ($d = 2$ or 3) with a piecewise smooth boundary $\partial\Omega$. We restrict ourselves to considering a square/cubic domain Ω under the Dirichlet boundary conditions. A fixed final time is denoted by T. The model treated here is of the form

$$\begin{cases} \dfrac{\partial A_1}{\partial t} = d_1\nabla^2 A_1 + A_1(r_1 - a_{11}A_1 - a_{12}A_2 - a_{13}A_3), & \text{in } \Omega\times(0,T], \\ \dfrac{\partial A_2}{\partial t} = d_2\nabla^2 A_2 + A_2(r_2 - a_{21}A_2 - a_{22}A_3 - a_{23}A_1), & \text{in } \Omega\times(0,T], \\ \dfrac{\partial A_3}{\partial t} = d_3\nabla^2 A_3 + A_3(r_3 - a_{31}A_3 - a_{32}A_1 - a_{33}A_2), & \text{in } \Omega\times(0,T], \end{cases} \tag{1}$$

where

- t is the time(s) and $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x, y, z)$ is a function of 2D position or 3D position in the Cartesian coordinate system, respectively;

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- $A_i(\boldsymbol{x}, t)$, $1 \le i \le 3$, present the population density of the ith species at $\boldsymbol{x}$ and t;
- d_i, $1 \le i \le 3$, are the positive diffusion coefficients;
- r_i, $1 \le i \le 3$, are the positive intrinsic growth rates of ith population;
- a_{ij}, $1 \le i, j \le 3$, are the positive coefficients accounting for the intra-specific competition if $i = j$, and the inter-specific competition if $i \ne j$.

Dirichlet boundary conditions on the three competing species are:

$$A_i(\boldsymbol{x}, t) = 0, \quad 1 \le i \le 3, \text{ on } \partial\Omega \times [0, T]. \tag{2}$$

Initial conditions are:

$$A_i(\boldsymbol{x}, t = 0) = A_{i,0}(\boldsymbol{x}), \quad 1 \le i \le 3, \boldsymbol{x} \in \bar{\Omega}, \tag{3}$$

where $A_{i,0}$, $1 \le i \le 3$, are the three prescribed competing species values; $\bar{\Omega} = \Omega \cup \partial\Omega$ will denote the closure of Ω.

The diffusive 3-species Lotka–Volterra (LV) systems as discussed by several investigators, e.g., [1–3] have been an active field of inquiry that mimic the population dynamics of interacting species because of their applications in the predator–prey system and mathematical biological models. The positive coefficients a_{ij}, $1 \le i, j \le 3$ play an important role in characterizing the 3-species competitive systems. Kishimoto [4] studied a stable non-constant equilibrium solution of the diffusive Lotka–Volterra system with three species. Kan-on [5] studied the existence and instability of Neumann layer solutions for a 3-component Lotka–Volterra model with diffusion. The effect of diffusion for the multispecies Lotka–Volterra competition model was investigated by Martínez [6]. For the cross-diffusion case, see [7].

It is worth mentioning that the diffusive 3-species Lotka–Volterra systems presented here have a tie to the classical Gause–Lotka–Volterra equations that consist of a set of time-dependent ordinary differential equations with nonlinear quadratic terms, e.g. [8,9]. Because of the sign of a_{ij}, $1 \le i, j \le 3$, Frachebourg et al. [10] investigated the spatial organization in cyclic Lotka–Volterra systems. Gyllenberg et al. [11,12] also studied limit cycles for competitor–competitor–mutualist Lotka–Volterra systems.

In this paper, we construct the semi-implicit finite element (FE) scheme for (1)–(3) in Section 2. The solvability of the proposed scheme will be demonstrated in Section 3. We make use of theory from prior estimates and techniques. Optimal order estimates in the H^1 norm are derived for the errors in the approximate solution. Error estimates will be given in Section 4. A second-order semi-implicit scheme in time for the LV equations will be given in Section 5. In Section 6, a series of numerical computations are carried out to examine the expected orders of accuracy in the error estimates. A few remarks will be given in Section 7.

2. Weak formulation and finite element formulation

2.1. Weak formulation

To obtain the weak formulation for the problem (1)–(3), we introduce some notation. We denote by $H^k(\Omega)$ the usual Sobolev space containing functions having the finite norm

$$\|u\|_k = \left[\int_\Omega \sum_{\alpha \le k} \left| D^\alpha u \right|^2 \mathrm{d}\boldsymbol{x} \right]^{1/2}.$$

Similarly,

$$\|u\|_\infty = \sup_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})|.$$

Let

$$H_0^1(\Omega) = \{u | u = 0 \text{ on } \partial\Omega\}.$$

We use the symbol $(u, v) := \int_\Omega uv d\boldsymbol{x}$ for the inner product on $L^2(\Omega) = H^0(\Omega)$ and

$$(\nabla u, \nabla v) := \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \mathrm{d}\boldsymbol{x}$$

for the scalar functions.

In the following, we use C to denote a generic positive constant which is independent of the mesh parameter h and the time-step Δt, and whose value is not necessarily the same on each occurrence. For a sequence $\{w^n(\boldsymbol{x})\}$ of functions associated with different time levels $n = 1, 2, \ldots, N$ with $N = T/\Delta t$, we adopt the notations

$$|||w|||_{\infty,k} := \max_{1\le n\le N} \|w^n\|_k,$$

and

$$|||w|||_{0,k} := \left[\sum_{n=1}^{n=N} \|w^n\|_k^2 \Delta t \right]^{\frac{1}{2}}.$$

When $w(\boldsymbol{x}, t)$ is defined on the whole time interval $(0, T)$, we use the analogous notations

$$\|w\|_{\infty,k} := \sup_{1<t<T} \|w(\cdot, t)\|_k,$$

and

$$\|w\|_{0,k} := \left[\int_0^T \|w(\cdot, t)\|_k^2 \mathrm{d}t \right]^{\frac{1}{2}}.$$

Let $V = H_0^1$. Multiplying each equation in (1) by the corresponding component of q and integrating each required equation on Ω, we have

$$\begin{cases} \left(\dfrac{\partial A_1}{\partial t}, q\right) + d_1 (\nabla A_1, \nabla q) = (A_1(r_1 - a_{11}A_1 - a_{12}A_2 - a_{13}A_3), q), & \forall q \in V, \\ \left(\dfrac{\partial A_2}{\partial t}, q\right) + d_2 (\nabla A_2, \nabla q) = (A_2(r_2 - a_{21}A_2 - a_{22}A_3 - a_{23}A_1), q), & \forall q \in V, \\ \left(\dfrac{\partial A_3}{\partial t}, q\right) + d_3 (\nabla A_3, \nabla q) = (A_3(r_3 - a_{31}A_3 - a_{32}A_1 - a_{33}A_2), q), & \forall q \in V. \end{cases} \tag{4}$$

We assume that the exact solution of problem (1) satisfies: for $i = 1, 2, 3$,

$$\|A_i\|_\infty, \|A_{i,t}\|_\infty, \|\nabla A_i\|_\infty \le M, \tag{5}$$

for $t \in [0, T]$, where M is constant and $A_{i,t} = \frac{\partial A_i}{\partial t}$.

A few well-known inequalities shall be used as follows:

E1: Let $0 \le a, b \in \mathbb{R}$, and then

$$2(a - b)a = a^2 - b^2 + (a - b)^2 \ge a^2 - b^2.$$

E2 (or Arithmetic–Geometric Mean Inequality (AGMI)): Let $a, b, \epsilon \in \mathbb{R}$ and $\epsilon > 0$, and then

$$ab \le \frac{1}{4\epsilon}a^2 + \epsilon b^2.$$

E3 (or Hölder Inequality (HI)): Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, where $p, q \le 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, and then

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

Recall that the Cauchy–Schwarz inequality (CSI) is just a special case of the HI when $p = q = 2$.

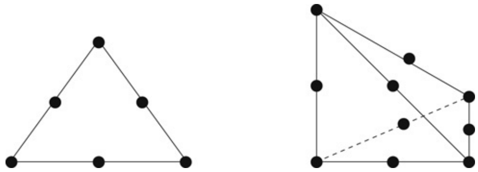

Fig. 1. Triangular/tetrahedral finite elements used: $\mathbb{P}_2$ for A_i, $i = 1$ to 3.

E4 (or Generalized Hölder Inequality (GHI)): Let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$ where $p, q, r \leq 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, and then

$$\int_\Omega |f(x)g(x)h(x)|\,\mathrm{d}x \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}.$$

Typical values of p, q, r are $p = q = 2$ and $r = \infty$.

2.2. Finite element formulation

Let $\mathcal{T}_h$ be a partition consisting of a regular isoparametric family of n-simplices with diameter not greater than h. Let $W_h \subset H_0^1$ denote the finite dimensional subspaces of piecewise polynomials of degree k associated with $\mathcal{T}_h$ and $V_h = W_h$. In this paper, we restrict ourselves to using the same approximation space for the three competing species. We shall use the quadratic finite element approximation. If $\mathcal{T}_h$ is the triangulation over Ω and $\bar{\Omega} = \cup\{\widetilde{K} : \widetilde{K} \in \bar{\mathcal{T}}_h\}$, the quadratic FE spaces can be defined as

$$W_h = \{w \in C^0(\bar{\Omega}) : w|_{\widetilde{K}} \in \mathbb{P}_2(\widetilde{K}), \forall \widetilde{K} \in \mathcal{T}_h\}.$$

Here, the space $\mathbb{P}_q(\widetilde{K})$ denotes the sets of polynomials of degree q in $\widetilde{K}$, i.e., for $q \geq 0$:

$$\mathbb{P}_q(\widetilde{K}) := \left\{ w : \widetilde{K} \to \mathbb{R}, w(x, y) = \sum_{0 \leq i+j \leq q} \alpha_{ij} x^i y^j \right\}.$$

Here, $\widetilde{K}$ denotes the triangular or the tetrahedral element. Their respective shape functions are used as illustrated in Fig. 1.

From the properties of the piecewise polynomial subspaces and the assumptions on the partition $\mathcal{T}_h$, we have the following approximation properties (AR):

$$\begin{cases} \inf\limits_{q \in V_h} \|z - q\|_0 \leq C h^{k+1} \|z\|_{k+1}, & \forall z \in V \cap H^{k+1}(\Omega), \\ \inf\limits_{q \in V_h} \|z - q\|_1 \leq C h^{k} \|z\|_{k+1}, & \forall z \in V \cap H^{k+1}(\Omega), \end{cases} \tag{6}$$

where the C's are constants independent of h; $k \geq 2$ is an integer. By the assumption of regular iso-parametric finite elements, we have the following inverse properties (IP):

$$\|q_h\|_\infty \leq C h^{-d/2} \|q_h\|_0, \quad \forall q_h \in V_h. \tag{7}$$

Let Δt be the time-step size for the time variable t and $t_n = n\Delta t$, $n = 0, 1, \ldots, N$ ($N = \frac{T}{\Delta t}$). We denote the backward difference quotient by $d_t f := \frac{f(t_n) - f(t_{n-1})}{\Delta t}$. To solve the LV equations numerically, the time-derivatives are replaced by backward difference quotients and the non-linear terms are linearized, while the diffusion terms are treated implicitly. At each time level, we solve a set of nonlinear parabolic-like evolution systems associated with the conforming FE method. This procedure results in the following formulation of the FE approximation to (1).

Problem A. For $n = 1, 2, \ldots, N$, find $(A^n_{1,h}, A^n_{2,h}, A^n_{3,h}) \in V_h \times V_h \times V_h$ such that

$$\begin{cases} (d_t A^n_{1,h}, q_1) + d_1(\nabla A^n_{1,h}, \nabla q_1) - r_1\left(A^{n-1}_{1,h}, q_1\right) - \left(A^n_{1,h}(-a_{11}A^{n-1}_{1,h} - a_{12}A^{n-1}_{2,h} - a_{13}A^{n-1}_{3,h}), q_1\right) = 0, \\ \quad \forall q_1 \in V_h, \\ (d_t A^n_{2,h}, q_2) + d_2(\nabla A^n_{2,h}, \nabla q_2) - r_2\left(A^{n-1}_{2,h}, q_2\right) - \left(A^n_{2,h}(-a_{21}A^{n-1}_{2,h} - a_{22}A^{n-1}_{3,h} - a_{23}A^{n-1}_{1,h}), q_2\right) = 0, \\ \quad \forall q_2 \in V_h, \\ (d_t A^n_{3,h}, q_3) + d_3(\nabla A^n_{3,h}, \nabla q_3) - r_3\left(A^{n-1}_{3,h}, q_3\right) - \left(A^n_{3,h}(-a_{31}A^{n-1}_{3,h} - a_{32}A^{n-1}_{1,h} - a_{33}A^{n-1}_{2,h}), q_3\right) = 0, \\ \quad \forall q_3 \in V_h. \end{cases} \tag{8}$$

Notice that (8) is a linear system for $(A^n_{1,h}, A^n_{2,h}, A^n_{3,h})$. This scheme is simple since the three linear systems of equations, $(A^n_{1,h}, A^n_{2,h}, A^n_{3,h})$ can be solved separately for each n.

We now focus on the analysis of the FE formulation (8). In our analysis, we assume the following induction hypothesis (IH-1), which shall be proved at the end of Section 4:

$$\|A^{n-1}_{i,h}\|_\infty \le K, \quad i = 1, 2, 3, \tag{9}$$

where K is constant.

3. Solvability of the proposed scheme

To ensure the computability of the proposed scheme, we shall prove the uniqueness and existence of the solution to Problem A (cf. (8)) for each n.

Find $(A^n_{1,h}, A^n_{2,h}, A^n_{3,h}) \in V_h \times V_h \times V_h$ such that

$$\widehat{a}(A^n_{1,h}, A^n_{2,h}, A^n_{3,h}; q_1, q_2, q_3) = \widehat{F}(q_1, q_2, q_3), \quad \forall q_i \in V_h, i = 1, 2, 3, \tag{10}$$

where

$$\begin{aligned} &\widehat{a}(A^n_{1,h}, A^n_{2,h}, A^n_{3,h}; q_1, q_2, q_3) \\ &\quad = \sum_{i=1}^{3}\left(\frac{A^n_{i,h}}{\Delta t}, q_i\right) + \sum_{i=1}^{3} d_i\left(\nabla A^n_{i,h}, \nabla q_i\right) - \left(A^n_{1,h}(-a_{11}A^{n-1}_{1,h} - a_{12}A^{n-1}_{2,h} - a_{13}A^{n-1}_{3,h}), q_1\right) \\ &\qquad - \left(A^n_{2,h}(-a_{21}A^{n-1}_{2,h} - a_{22}A^{n-1}_{3,h} - a_{23}A^{n-1}_{1,h}), q_2\right) - \left(A^n_{3,h}(-a_{31}A^{n-1}_{3,h} - a_{32}A^{n-1}_{1,h} - a_{33}A^{n-1}_{2,h}), q_3\right) \end{aligned} \tag{11}$$

and

$$\widehat{F}(q_1, q_2, q_3) = \sum_{i=1}^{3}\left(\frac{A^{n-1}_{i,h}}{\Delta t}, q_i\right) + \sum_{i=1}^{3} r_i\left(A^{n-1}_{i,h}, q_i\right). \tag{12}$$

Note that (10) is a linear system of finite equations. The uniqueness of the solution to this system implies the existence of the solution. Therefore, we only need to prove that there exists at most one solution. This can be achieved by proving the positivity of the bilinear form $\widehat{a}$. To do this, we consider

$$\begin{aligned} &\widehat{a}(q_1, q_2, q_3; q_1, q_2, q_3) \\ &\quad = \sum_{i=1}^{3}\frac{1}{\Delta t}(q_i, q_i) + \sum_{i=1}^{3} d_i(\nabla q_i, \nabla q_i) - \left(q_1(-a_{11}A^{n-1}_{1,h} - a_{12}A^{n-1}_{2,h} - a_{13}A^{n-1}_{3,h}), q_1\right) \\ &\qquad - \left(q_2(-a_{21}A^{n-1}_{2,h} - a_{22}A^{n-1}_{3,h} - a_{23}A^{n-1}_{1,h}), q_2\right) - \left(q_3(-a_{31}A^{n-1}_{3,h} - a_{32}A^{n-1}_{1,h} - a_{33}A^{n-1}_{2,h}), q_3\right). \end{aligned} \tag{13}$$

Using the inequality (E4 or GHI) and applying the assumption (9), all the quadratic terms become

$$\left|\left(A^{n-1}_{i,h} q_1, q_1\right)\right| \le K\|q_1\|_0\|q_1\|_0 \le K\|q_1\|_0^2,$$
$$\left|\left(A^{n-1}_{i,h} q_2, q_2\right)\right| \le K\|q_2\|_0^2$$

and

$$\left|\left(A_{i,h}^{n-1}q_3, q_3\right)\right| \leq K\|q_3\|_0^2.$$

Gathering all these estimates, we deduce from (13) that

$$\begin{aligned}
&\widehat{a}(q_1, q_2, q_3; q_1, q_2, q_3) \\
&\quad \geq \sum_{i=1}^{3} \frac{1}{\Delta t}\|q_i\|_0^2 + \sum_{i=1}^{3} d_i\|\nabla q_i\|_0^2 + \left(q_1(-a_{11}A_{1,h}^{n-1} - a_{12}A_{2,h}^{n-1} - a_{13}A_{3,h}^{n-1}), q_1\right) \\
&\qquad + \left(q_2(-a_{21}A_{2,h}^{n-1} - a_{22}A_{3,h}^{n-1} - a_{23}A_{1,h}^{n-1}), q_2\right) + \left(q_3(-a_{31}A_{3,h}^{n-1} - a_{32}A_{1,h}^{n-1} - a_{33}A_{2,h}^{n-1}), q_3\right) \\
&\quad \geq \sum_{i=1}^{3} \frac{1}{\Delta t}\|q_i\|_0^2 + \sum_{i=1}^{3} d_i\|\nabla q_i\|_0^2 + (-a_{11}K - a_{12}K - a_{13}K)\|q_1\|_0^2 \\
&\qquad + (-a_{21}K - a_{22}K - a_{23}K)\|q_2\|_0^2 + (-a_{31}K - a_{32}K - a_{33}K)\|q_3\|_0^2.
\end{aligned} \tag{14}$$

Choosing small time-step size Δt such that

$$\Delta t \leq \left\{\frac{1}{2(a_{11}+a_{12}+a_{13})K}, \frac{1}{2(a_{21}+a_{22}+a_{23})K}, \frac{1}{2(a_{31}+a_{32}+a_{33})K}\right\}, \tag{15}$$

we have

$$\widehat{a}(q_1, q_2, q_3; q_1, q_2, q_3) \geq \sum_{i=1}^{3} \frac{1}{2\Delta t}\|q_i\|_0^2 + \sum_{i=1}^{3} d_i\|\nabla q_i\|_0^2. \tag{16}$$

Therefore, we have the following result:

Lemma 1. *Assume (9) is true. For a sufficiently small time-step size Δt, there exists a unique solution $(A_{1,h}^n, A_{2,h}^n, A_{3,h}^n) \in V_h \times V_h \times V_h$ satisfying (8).*

4. Error analysis

In this section, we shall prove the following *a priori* error estimate of the numerical solution. We start by working with a similar procedure to Liu's [13,14] for the case of compressible Navier–Stokes flow problems. We assume that the initial conditions by choosing $A_{i,h}^0$ as the Ritz projection of A_i^0 onto the FE subspace V_h to the numerical algorithm (8) is considered.

Theorem 2. *Suppose (5) holds. Suppose V_h is a subspace of piecewise polynomials of degree k with $k > 1$. There is a constant $C > 0$ such if $\Delta t \leq Ch^{d/2}$, the numerical solution of (8) is convergent to the exact solution of (1) as $h \to 0$. Moreover, letting $(A_{1,h}^n, A_{2,h}^n, A_{3,h}^n)$ and (A_1, A_2, A_3) be the numerical solution and the exact solution, respectively, we have the following error estimates:*

$$\begin{cases} |||A_{i,h} - A_i|||_{\infty,0} \leq C_1 h^k + C_2 \Delta t, \\ |||A_{i,h} - A_i|||_{0,1} \leq C_1 h^k + C_2 \Delta t, \end{cases} \tag{17}$$

where C_i's are independent of h and Δt, and they depend on the exact solution (A_1, A_2, A_3) as follows:

$$\begin{aligned}
C_1 &= C\left(|||A_i|||_{0,k+1} + |||A_{i,t}|||_{0,k}\right), \\
C_2 &= C\left(\|A_{i,t}\|_{\infty,0} + \|A_{i,tt}\|_{0,0}\right).
\end{aligned}$$

Proof. Let $A_i^n = A_i(t_n)$ be the exact solution of (1). Let a_i^n be the Ritz projection of A_i^n onto V_h satisfying

$$\begin{cases} \|a_i^n - A_i^n\|_0 \leq Ch^{k+1}\|A_i\|_{k+1}, \\ \|\nabla(a_i^n - A_i^n)\|_1 \leq Ch^k\|A_i\|_{k+1}. \end{cases} \tag{18}$$

Let

$$\theta_i^n = A_i^n - a_i^n, \qquad \eta_i^n = a_i^n - A_{i,h}^n,$$

where $A_{i,h}^n$ are numerical solutions of (8). The terms θ_i^n can be considered as spatial approximation errors, while η_i^n are consistency errors induced by the combination of the time-stepping and the space approximation.

Denote

$$e_{A_i}^n = A_i^n - A_{i,h}^n = \theta_i^n + \eta_i^n. \tag{19}$$

To estimate $e_{A_i}^n$, we only need to estimate η_i^n since bounds for θ_i^n are given in (18), depending on the degrees of piecewise polynomial spaces. We assume another induction hypothesis, which will be verified at the end of this section:

$$\sum_{k=1}^{n-1} \Delta t \|\nabla \eta_i^k\|_\infty \le 1. \tag{20}$$

From (4), the exact solution satisfies the following equations:

$$\begin{cases} \left(\dfrac{\partial A_1^n}{\partial t}, q_1\right) + d_1(\nabla A_1^n, \nabla q_1) - r_1\left(A_1^n, q_1\right) - \left(A_1^n(-a_{11}A_1^n - a_{12}A_2^n - a_{13}A_3^n), q_1\right) = 0, \\ \quad \forall q_1 \in V_h, \\ \left(\dfrac{\partial A_2^n}{\partial t}, q_2\right) + d_2(\nabla A_2^n, \nabla q_2) - r_2\left(A_2^n, q_2\right) - \left(A_2^n(-a_{21}A_2^n - a_{22}A_3^n - a_{23}A_1^n), q_2\right) = 0, \\ \quad \forall q_2 \in V_h, \\ \left(\dfrac{\partial A_3^n}{\partial t}, q_3\right) + d_3(\nabla A_3^n, \nabla q_3) - r_3\left(A_3^n, q_3\right) - \left(A_3^n(-a_{31}A_3^n - a_{32}A_1^n - a_{33}A_2^n), q_3\right) = 0, \\ \quad \forall q_3 \in V_h. \end{cases} \tag{21}$$

Subtracting (8) from (21), we obtain the following equations for $e_{A_i}^n$:

$$\begin{cases} \left(d_t e_{A_1}^n, q_1\right) + d_1(\nabla e_{A_1}^n, \nabla q_1) = \left(d_t A_1^n - \dfrac{\partial A_1^n}{\partial t}, q_1\right) + r_1\left(A_1^n - A_{1,h}^{n-1}, q_1\right) + \Big(A_1^n(-a_{11}(A_1^n - A_{1,h}^{n-1}) \\ \quad - a_{12}(A_2^n - A_{2,h}^{n-1}) - a_{13}(A_3^n - A_{3,h}^{n-1})), q_1\Big) + \Big(-a_{11}(A_1^n - A_{1,h}^n)A_{1,h}^{n-1} \\ \quad - a_{12}(A_1^n - A_{1,h}^n)A_{2,h}^{n-1} - a_{13}(A_1^n - A_{1,h}^n)A_{3,h}^{n-1}, q_1\Big) = 0, \quad \forall q_1 \in V_h, \\ \left(d_t e_{A_2}^n, q_2\right) + d_2(\nabla e_{A_2}^n, \nabla q_2) = \left(d_t A_2^n - \dfrac{\partial A_2^n}{\partial t}, q_2\right) + r_2\left(A_2^n - A_{2,h}^{n-1}, q_2\right) + \Big(A_2^n(-a_{21}(A_2^n - A_{2,h}^{n-1}) \\ \quad - a_{22}(A_3^n - A_{3,h}^{n-1}) - a_{23}(A_1^n - A_{1,h}^{n-1})), q_2\Big) + \Big(-a_{21}(A_2^n - A_{2,h}^n)A_{2,h}^{n-1} \\ \quad - a_{22}(A_2^n - A_{2,h}^n)A_{3,h}^{n-1} - a_{23}(A_2^n - A_{2,h}^n)A_{1,h}^{n-1}, q_2\Big) = 0, \quad \forall q_2 \in V_h, \\ \left(d_t e_{A_3}^n, q_3\right) + d_3(\nabla e_{A_3}^n, \nabla q_3) = \left(d_t A_3^n - \dfrac{\partial A_3^n}{\partial t}, q_3\right) + r_3\left(A_3^n - A_{3,h}^{n-1}, q_3\right) + \Big(A_3^n(-a_{31}(A_3^n - A_{3,h}^{n-1}) \\ \quad - a_{32}(A_1^n - A_{1,h}^{n-1}) - a_{33}(A_2^n - A_{2,h}^{n-1})), q_3\Big) + \Big(-a_{31}(A_3^n - A_{3,h}^n)A_{3,h}^{n-1} \\ \quad - a_{32}(A_3^n - A_{3,h}^n)A_{1,h}^{n-1} - a_{33}(A_3^n - A_{3,h}^n)A_{2,h}^{n-1}, q_3\Big) = 0, \quad \forall q_3 \in V_h. \end{cases} \tag{22}$$

Substituting (19) into (22) and taking $q_i = \eta_i^n$, we obtain

$$(d_t \eta_i^n, \eta_i^n) + d_i(\nabla \eta_i^n, \nabla \eta_i^n) = -\left(d_t \theta_i^n, \eta_i^n\right) - d_i(\nabla \theta_i^n, \nabla \eta_i^n) + R_i(\eta_i^n), \tag{23}$$

where

$$\begin{cases} R_1(\eta_1^n) = \left(d_t A_1^n - \dfrac{\partial A_1^n}{\partial t}, \eta_1^n\right) + r_1\left(A_1^n - A_{1,h}^{n-1}, \eta_1^n\right) + \Big(A_1^n(-a_{11}(A_1^n - A_{1,h}^{n-1}) - a_{12}(A_2^n - A_{2,h}^{n-1}) \\ \quad - a_{13}(A_3^n - A_{3,h}^{n-1})), \eta_1^n\Big) + \Big(-a_{11}(A_1^n - A_{1,h}^n)A_{1,h}^{n-1} \\ \quad - a_{12}(A_1^n - A_{1,h}^n)A_{2,h}^{n-1} - a_{13}(A_1^n - A_{1,h}^n)A_{3,h}^{n-1}, \eta_1^n\Big), \\ R_2(\eta_2^n) = \left(d_t A_2^n - \dfrac{\partial A_2^n}{\partial t}, \eta_2^n\right) + r_2\left(A_2^n - A_{2,h}^{n-1}, \eta_2^n\right) + \Big(A_2^n(-a_{21}(A_2^n - A_{2,h}^{n-1}) - a_{22}(A_3^n - A_{3,h}^{n-1}) \\ \quad - a_{23}(A_1^n - A_{1,h}^{n-1})), \eta_2^n\Big) + \Big(-a_{21}(A_2^n - A_{2,h}^n)A_{2,h}^{n-1} \\ \quad - a_{22}(A_2^n - A_{2,h}^n)A_{3,h}^{n-1} - a_{23}(A_2^n - A_{2,h}^n)A_{1,h}^{n-1}, \eta_2^n\Big), \\ R_3(\eta_3^n) = \left(d_t A_3^n - \dfrac{\partial A_3^n}{\partial t}, \eta_3^n\right) + r_3\left(A_3^n - A_{3,h}^{n-1}, \eta_3^n\right) + \Big(A_3^n(-a_{31}(A_3^n - A_{3,h}^{n-1}) - a_{32}(A_1^n - A_{1,h}^{n-1}) \\ \quad - a_{33}(A_2^n - A_{2,h}^{n-1})), \eta_3^n\Big) + \Big(-a_{31}(A_3^n - A_{3,h}^n)A_{3,h}^{n-1} \\ \quad - a_{32}(A_3^n - A_{3,h}^n)A_{1,h}^{n-1} - a_{33}(A_3^n - A_{3,h}^n)A_{2,h}^{n-1}, \eta_3^n\Big). \end{cases} \tag{24}$$

Applying the inequality (E1), (23) yields

$$\frac{1}{2\Delta t}\sum_{i=1}^{3}\left(\|\eta_i^n\|_0^2 - \|\eta_i^{n-1}\|_0^2\right) + \sum_{i=1}^{3} d_i\|\nabla\eta_i^n\|_0^2 \le \sum_{i=1}^{3}\left|\left(d_t\theta_i^n, \eta_i^n\right)\right| + \sum_{i=1}^{3}\left|d_i(\nabla\theta_i^n, \nabla\eta_i^n)\right| + \sum_{i=1}^{3}\left|R_i(\eta_i^n)\right|. \tag{25}$$

Multiplying (25) by Δt and summing from $n = 1$ to l, we have

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{3}\left(\|\eta_i^l\|_0^2 - \|\eta_i^0\|_0^2\right) + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3} d_i\|\nabla\eta_i^n\|_0^2 \\ &\quad \le \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\left|\left(d_t\theta_i^n, \eta_i^n\right)\right| + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\left|d_i(\nabla\theta_i^n, \nabla\eta_i^n)\right| + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\left|R_i(\eta_i^n)\right|. \end{aligned} \tag{26}$$

Now we will discuss and simplify (26) in a term-by-term manner. In the following estimation, (5), AP (cf. (6)), IP (cf. (7)), IH-1 (cf. (9)), (18) and the inequalities (E2 and E3 or AGMI and CSI), will be used frequently.

Since $A_{i,h}^{n-1}|_{\partial\Omega} = 0$, after applying the Green theorem, for the third and fourth terms of the right-hand side of (26), we have

- $$\left|\left(d_t\theta_i^n, \eta_i^n\right)\right| \le \frac{1}{4}\|\eta_i^n\|_0^2 + \|d_t\theta_i^n\|_0^2, \tag{27}$$
- $$\left|d_i\left(\nabla\theta_i^n, \nabla\eta_i^n\right)\right| \le d_i\left(\frac{1}{4}\|\nabla\eta_i^n\|_0^2 + \|\nabla\theta_i^n\|_0^2\right). \tag{28}$$

For $R_i(\eta_i^n)$, we have

- for $i = 1, 2, 3$,

$$\begin{aligned} \left|r_i\left(A_i^n - A_{i,h}^{n-1}, \eta_i^n\right)\right| &\le r_i\left(\|\eta_i^n\|_0\|A_i^n - A_{i,h}^{n-1}\|_0\right) \\ &\le r_i\left(\frac{1}{2}\|\eta_i^n\|_0^2 + \frac{1}{2}\|A_i^n - A_i^{n-1} + A_i^{n-1} - a_i^{n-1} + a_i^{n-1} - A_{i,h}^{n-1}\|_0^2\right) \\ &\le r_i\left(\frac{1}{2}\|\eta_i^n\|_0^2 + \frac{1}{4}\left(\|\theta_i^{n-1}\|_0^2 + \|\eta_i^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{i,t}\|_{\infty,0}^2\right), \end{aligned} \tag{29}$$

- for $i = 1, 2, 3$,

$$\left|\left(A_i^n(-a_{i1}(A_i^n - A_{i,h}^{n-1})), \eta_i^n\right)\right| \le a_{i1}M\left(\|\eta_i^n\|_0\|A_i^n - A_{i,h}^{n-1}\|_0\right)$$
$$\le a_{i1}M\left(\frac{1}{2}\|\eta_i^n\|_0^2 + \frac{1}{4}\left(\|\theta_i^{n-1}\|_0^2 + \|\eta_i^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{i,t}\|_{\infty,0}^2\right), \tag{30}$$

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$$\left|\left(A_1^n(-a_{12}(A_2^n - A_{2,h}^{n-1})), \eta_1^n\right)\right| \le a_{12}M\left(\frac{1}{2}\|\eta_1^n\|_0^2 + \frac{1}{4}\left(\|\theta_2^{n-1}\|_0^2 + \|\eta_2^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{2,t}\|_{\infty,0}^2\right), \tag{31}$$

$$\left|\left(A_2^n(-a_{22}(A_3^n - A_{3,h}^{n-1})), \eta_2^n\right)\right| \le a_{22}M\left(\frac{1}{2}\|\eta_2^n\|_0^2 + \frac{1}{4}\left(\|\theta_3^{n-1}\|_0^2 + \|\eta_3^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{3,t}\|_{\infty,0}^2\right), \tag{32}$$

$$\left|\left(A_3^n(-a_{32}(A_1^n - A_{1,h}^{n-1})), \eta_3^n\right)\right| \le a_{32}M\left(\frac{1}{2}\|\eta_3^n\|_0^2 + \frac{1}{4}\left(\|\theta_1^{n-1}\|_0^2 + \|\eta_1^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{1,t}\|_{\infty,0}^2\right), \tag{33}$$

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$$\left|\left(A_1^n(-a_{13}(A_3^n - A_{3,h}^{n-1})), \eta_1^n\right)\right| \le a_{13}M\left(\frac{1}{2}\|\eta_1^n\|_0^2 + \frac{1}{4}\left(\|\theta_3^{n-1}\|_0^2 + \|\eta_3^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{3,t}\|_{\infty,0}^2\right), \tag{34}$$

$$\left|\left(A_2^n(-a_{23}(A_1^n - A_{1,h}^{n-1})), \eta_2^n\right)\right| \le a_{23}M\left(\frac{1}{2}\|\eta_2^n\|_0^2 + \frac{1}{4}\left(\|\theta_1^{n-1}\|_0^2 + \|\eta_1^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{1,t}\|_{\infty,0}^2\right), \tag{35}$$

$$\left|\left(A_3^n(-a_{33}(A_2^n - A_{2,h}^{n-1})), \eta_3^n\right)\right| \le a_{33}M\left(\frac{1}{2}\|\eta_3^n\|_0^2 + \frac{1}{4}\left(\|\theta_2^{n-1}\|_0^2 + \|\eta_2^{n-1}\|_0^2\right) + \frac{1}{4}(\Delta t)^2\|A_{2,t}\|_{\infty,0}^2\right), \tag{36}$$

- for $i = 1, 2, 3$,

$$\left|\left(-a_{i1}(A_i^n - A_{i,h}^n)A_{i,h}^{n-1}, \eta_i^n\right)\right| \le a_{i1}K\left(\|\eta_i^n\|_0\|A_i^n - A_{i,h}^n\|_0\right)$$
$$\le a_{i1}K\left(\frac{1}{2}\|\eta_i^n\|_0^2 + \frac{1}{2}\|\theta_i^n\|_0^2 + \|\eta_i^n\|_0^2\right), \tag{37}$$

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$$\left|\left(-a_{12}(A_1^n - A_{1,h}^n)A_{2,h}^{n-1}, \eta_1^n\right)\right| \le a_{12}K\left(\frac{1}{2}\|\eta_1^n\|_0^2 + \frac{1}{2}\|\theta_1^n\|_0^2 + \|\eta_1^n\|_0^2\right), \tag{38}$$

$$\left|\left(-a_{22}(A_2^n - A_{2,h}^n)A_{3,h}^{n-1}, \eta_2^n\right)\right| \le a_{22}K\left(\frac{1}{2}\|\eta_2^n\|_0^2 + \frac{1}{2}\|\theta_2^n\|_0^2 + \|\eta_2^n\|_0^2\right), \tag{39}$$

$$\left|\left(-a_{32}(A_3^n - A_{3,h}^n)A_{1,h}^{n-1}, \eta_3^n\right)\right| \le a_{32}K\left(\frac{1}{2}\|\eta_3^n\|_0^2 + \frac{1}{2}\|\theta_3^n\|_0^2 + \|\eta_3^n\|_0^2\right), \tag{40}$$

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$$\left|\left(-a_{13}(A_1^n - A_{1,h}^n)A_{3,h}^{n-1}, \eta_1^n\right)\right| \le a_{13}K\left(\frac{1}{2}\|\eta_1^n\|_0^2 + \frac{1}{2}\|\theta_1^n\|_0^2 + \|\eta_1^n\|_0^2\right), \tag{41}$$

$$\left|\left(-a_{23}(A_2^n - A_{2,h}^n)A_{1,h}^{n-1}, \eta_2^n\right)\right| \le a_{23}K\left(\frac{1}{2}\|\eta_2^n\|_0^2 + \frac{1}{2}\|\theta_2^n\|_0^2 + \|\eta_2^n\|_0^2\right), \tag{42}$$

$$\left|\left(-a_{33}(A_3^n - A_{3,h}^n)A_{2,h}^{n-1}, \eta_3^n\right)\right| \le a_{33}K\left(\frac{1}{2}\|\eta_3^n\|_0^2 + \frac{1}{2}\|\theta_3^n\|_0^2 + \|\eta_3^n\|_0^2\right). \tag{43}$$

By setting $\eta_i^0 = 0$, we choose the initial conditions $A_{i,h}^0 = A_i^0$. Applying the above inequalities to (26), we get

$$\frac{1}{2}\sum_{i=1}^{3}\|\eta_i^l\|_0^2 + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\frac{d_i}{(4/3)}\|\nabla\eta_i^n\|_0^2$$
$$\le \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}d_i\|\nabla\theta_i^n\|_0^2 + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\Pi\left(\|\theta_i^n\|_0^2 + \|\eta_i^n\|_0^2\right) + \Delta t\sum_{n=1}^{n=l}\sum_{i=1}^{3}\|d_t\theta_i^n\|_0^2$$

$$+ \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \Pi \left(\|\theta_i^{n-1}\|_0^2 + \|\eta_i^{n-1}\|_0^2 \right) + CT\, |\Delta t|^2 \sum_{i=1}^{3} \|A_{i,t}\|_{\infty,0}^2$$

$$+ \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \left| \left(d_t A_i^n - \frac{\partial A_i}{\partial t}, \eta_i^n \right) \right|, \tag{44}$$

where $\Pi = \Pi(M, K, r_i, a_{ij})$, $1 \le i, j \le 3$. We now estimate the third, fourth, fifth, sixth and eighth terms on the right-hand side of (44). On the basis of the discrete Gronwall lemma, the error estimate theorem will be obtained.

- Using the properties of the approximation spaces (cf. (6) and (18)), we have

$$\sum_{n=1}^{n=l} \Delta t \|\nabla \theta_i^n\|_0^2 \le Ch^{2k} \sum_{n=1}^{n=l} \Delta t \|A_i^n\|_{k+1}^2 \le Ch^{2k} |||A_i|||_{0,k+1}^2, \tag{45}$$

  and

$$\sum_{n=1}^{n=l} \Delta t \|\theta_i^n\|_0^2 \le Ch^{2k+2} \sum_{n=1}^{n=l} \Delta t \|A_i^n\|_{k+1}^2 \le Ch^{2k+2} |||A_i|||_{0,k+1}^2. \tag{46}$$

- Using the properties of the approximation spaces (cf. (6) and (18)), we have

$$\begin{aligned} \sum_{n=1}^{n=l} \Delta t \|d_t \theta_i^n\|_0^2 &\le \sum_{n=1}^{n=l} \left\| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \frac{\partial \theta_i}{\partial t} \mathrm{d}\tau \right\|_0^2 \Delta t \\ &\le \sum_{n=1}^{n=l} \left\{ \frac{1}{(\Delta t)^2} \int_{t_{n-1}}^{t_n} \|1\|_0^2 \mathrm{d}\tau \right\} \left\{ \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \theta_i}{\partial t} \right\|_0^2 \mathrm{d}\tau \right\} \Delta t \\ &\le \sum_{n=1}^{n=l} \frac{1}{(\Delta t)^2} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \theta_i}{\partial t} \right\|_0^2 \mathrm{d}\tau\, \Delta t\, \Delta t \\ &\le Ch^{2k} \|A_{i,t}\|_{0,k}^2. \end{aligned} \tag{47}$$

- For the last term on the right-hand side of (44), we apply the Taylor series with the remainder in terms of an integral:

$$\frac{f(t_n) - f(t_{n-1})}{\Delta t} \doteq \left. \frac{\partial f}{\partial t} \right|_{t_{n-1}} + \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} f_{tt}(\cdot, t)\, (t_{n-1} - t)\, \mathrm{d}t$$

  or

$$d_t f - f_t = \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} f_{tt}(\cdot, t)\, (t_{n-1} - t)\, \mathrm{d}t$$

  to $d_t A_i^n - A_{i,t}^n$. Using the inequality (E3), we have

$$\begin{aligned} \sum_{n=1}^{n=l} \Delta t \left(d_t A_i^n - A_{i,t}^n, \eta_i^n \right) &\le \sum_{n=1}^{n=l} \left[\int_\Omega \left\{ \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2 \mathrm{d}t \int_{t_{n-1}}^{t_n} A_{i,tt}^2(\cdot, t) \mathrm{d}t \right\} \mathrm{d}\boldsymbol{x} \right]^{1/2} \|\eta_i^n\|_0 \\ &\le \sum_{n=1}^{n=l} \frac{1}{\sqrt{3}} (\Delta t)^{3/2} \left[\int_\Omega \int_{t_{n-1}}^{t_n} A_{i,tt}^2(\cdot, t) \mathrm{d}t \mathrm{d}\boldsymbol{x} \right]^{1/2} \|\eta_i^n\|_0 \\ &\le \sum_{n=1}^{n=l} \Delta t \|\eta_i^n\|_0^2 + \frac{1}{3} (\Delta t)^2 |||A_{i,tt}|||_{0,0}^2. \end{aligned} \tag{48}$$

  Substituting the estimates (45)–(48) into (44), we get

$$\sum_{i=1}^{3} \|\eta_i^l\|_0^2 + \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \frac{d_i}{(4/3)} \|\nabla \eta_i^n\|_0^2$$

$$\leq \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \Pi \|\eta_i^n\|_0^2 + C\left(\beta h^{2k} + \Pi h^{2k+2}\right) \sum_{i=1}^{3} \left(\|\!|A_i|\!\|_{0,k+1} + \|A_{i,t}\|_{0,k}\right) + CT(\Delta t)^2$$
$$\times \sum_{i=1}^{3} \left(\|A_{i,t}\|_{\infty,0} + \|A_{i,tt}\|_{0,0}\right), \tag{49}$$

where C is a constant independent of l, Δt and h; it depends on d_i, r_i, a_{ij}, M, K, and $\beta = d_i$. By letting $h^2(\Pi/\beta) \leq 1$ and assuming that h is small, and denoting

$$\tilde{C}_1 = C \sum_{i=1}^{3} \left(\|\!|A_i|\!\|_{0,k+1} + \|A_{i,t}\|_{0,k}\right),$$
$$\tilde{C}_2 = C \sum_{i=1}^{3} \left(\|A_{i,t}\|_{\infty,0} + \|A_{i,tt}\|_{0,0}\right),$$

we have

$$\sum_{i=1}^{3} \|\eta_i^l\|_0^2 + \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \frac{d_i}{E} \|\nabla \eta_i^n\|_0^2 \leq \Delta t \sum_{n=1}^{n=l} \sum_{i=1}^{3} \Pi \|\eta_i^n\|_0^2 + \tilde{C}_1 h^{2k} + \tilde{C}_2 (\Delta t)^2, \tag{50}$$

where $E = 4/3$.

We summarize the results as follows:

1. Dropping the third and fourth terms on the right-hand side of (50), using the induction hypothesis (IH-2) in (20) and the discrete Gronwall Lemma, we have

$$\|\eta_i^n\|_0^2 \leq C\left(\tilde{C}_1 h^{2k} + \tilde{C}_2 (\Delta t)^2\right). \tag{51}$$

   Combining (18) and (51), we obtain

$$\|\!|A_i - A_{i,h}|\!\|_{\infty,0} \leq \|\!|\eta_i|\!\|_{\infty,0} + \|\!|\theta_i|\!\|_{\infty,0} \leq C_1 h^k + C_2 \Delta t$$

   which is nothing but the first estimate in Theorem 2.
2. Considering (51) and (50), we obtain

$$\|\!|\nabla \eta_i|\!\|_{0,0}^2 \leq CT\left(\tilde{C}_1 h^{2k} + \tilde{C}_2 (\Delta t)^2\right)$$

   and

$$\|\!|\eta_i|\!\|_{0,0}^2 = \sum_{n=1}^{n=N} \Delta t \left(\|\eta_i^n\|_0^2\right) \leq CT\left(\tilde{C}_1 h^{2k} + \tilde{C}_2 (\Delta t)^2\right).$$

   Therefore

$$\|\!|\eta_i|\!\|_{0,1}^2 \leq 2CT\left(\tilde{C}_1 h^{2k} + \tilde{C}_2 (\Delta t)^2\right)$$

   which is the second estimate in Theorem 2.

The convergence of the numerical solution results from the error estimates. The induction hypotheses (cf. (9) and (20)) need to be checked.

Proof of (9). Assume that (9) is true for $n = 1, 2, \ldots, l-1$.

•

$$\begin{aligned}
\|A_{i,h}^l\|_\infty &\leq \|A_{i,h}^l - A_i^l\|_\infty + \|A_i^l\|_\infty \quad \text{by the triangular inequality} \\
&\leq \|A_{i,h}^l - a_i^l + a_i^l - A_i^l\|_\infty + M \quad \text{by (9)} \\
&\leq \|\theta_i^l\|_\infty + \|\eta_i^l\|_\infty + M \quad \text{by the triangular inequality} \\
&\leq Ch^{-d/2}\|\theta_i^l\|_0 + Ch^{-d/2}\|\eta_i^l\|_0 + M \quad \text{by (7)}
\end{aligned}$$

$$\le Ch^{-d/2}\left(h^{k+1}\right) + Ch^{-d/2}\left(\Delta t + h^k\right) + M \quad \text{by (6) and (51)}$$
$$\le \underbrace{C\left(h^{k+1-d/2} + \frac{\Delta t}{h^{d/2}} + h^{k-d/2}\right)}_{[*]} + M.$$

First, we observe that the quantity $[*]$ is independent of l. For $k > d/2$ and a small h and Δt such that

$$h^{k-d/2} \le \frac{1}{C} \quad \text{and} \quad \Delta t \le \frac{h^{d/2}}{C}, \tag{52}$$

We then estimate

$$\|A^l_{i,h}\|_\infty \le 3 + M.$$

Therefore, the inductive hypothesis (IH-1) in (9) is true for $K \ge 3 + M$. □

Proof of (20). Assume (20) is true for $n = 1, 2, \ldots, l-1$.

- (50) and (51) yield

$$\sum_{n=1}^{n=l} \Delta t \|\nabla \eta_i^n\|_0^2 \le C\left(h^{2k} + (\Delta t)^2\right).$$

- Using (7), applying the inequality

$$\frac{1}{n}\sum_{k=1}^{n} a_k \le \sqrt{\frac{\sum_{k=1}^{n} a_k^2}{n}},$$

and replacing a_k by $\|\nabla \eta_i^k\|_0$, we have

$$\sum_{n=1}^{n=l} \Delta t \|\nabla \eta_i^n\|_\infty \le Ch^{-d/2} \sum_{n=1}^{n=l} \Delta t \|\nabla \eta_i^n\|_0$$
$$\le Ch^{-d/2}[\Delta t]^{1/2}\, l \sqrt{\frac{\sum_{n=1}^{n=l} \Delta t \|\nabla \eta_i^n\|_0^2}{l}}$$
$$\le \bar{C}\left(\frac{\Delta t}{h^{d/2}} + h^{k-d/2}\right),$$

where $\bar{C}$ is a constant independent of l, h, and Δt. Therefore, when

$$h^{k-d/2} \le \frac{1}{2\bar{C}} \quad \text{and} \quad \Delta t \le \frac{h^{d/2}}{2\bar{C}}, \tag{53}$$

(20) is true for n. □

In the proofs of the error estimates and the induction hypothesis, the only restrictions concerning the time-step size Δt are (15), (52) and (53). They are all consistent. Therefore, there exists a constant C such that $\Delta t \le Ch^{d/2}$, and the results in Theorem 2 are valid.

5. Second-order scheme

To obtain a second-order scheme in space and in time, we use a three-level backward differencing scheme and a linear extrapolation in time of the generation-like term and the non-linear term.

Problem B. For $n = 1$, we use the first-order scheme (cf. (8)). For $n = 2, \ldots, N$, find $(A^n_{1,h}, A^n_{2,h}, A^n_{3,h}) \in V_h \times V_h \times V_h$ such that

$$\begin{cases} \left(\dfrac{3A^n_{1,h} - 4A^{n-1}_{1,h} + A^{n-2}_{1,h}}{2\Delta t}, q_1\right) + d_1(\nabla A^n_{1,h}, \nabla q_1) - r_1\left(2A^{n-1}_{1,h} - A^{n-2}_{1,h}, q_1\right) - \Big(A^n_{1,h}(-a_{11}(2A^{n-1}_{1,h} \\ \quad -A^{n-2}_{1,h}) - a_{12}(2A^{n-1}_{2,h} - A^{n-2}_{2,h}) - a_{13}(2A^{n-1}_{3,h} - A^{n-2}_{3,h}), q_1)\Big) = 0, \quad \forall q_1 \in V_h, \\ \left(\dfrac{3A^n_{2,h} - 4A^{n-1}_{2,h} + A^{n-2}_{2,h}}{2\Delta t}, q_2\right) + d_2(\nabla A^n_{2,h}, \nabla q_2) - r_2\left(2A^{n-1}_{2,h} - A^{n-2}_{2,h}, q_2\right) - \Big(A^n_{2,h}(-a_{21}(2A^{n-1}_{2,h} \\ \quad -A^{n-2}_{2,h}) - a_{22}(2A^{n-1}_{3,h} - A^{n-2}_{3,h}) - a_{23}(2A^{n-1}_{1,h} - A^{n-2}_{1,h}), q_2)\Big) = 0, \quad \forall q_2 \in V_h, \\ \left(\dfrac{3A^n_{3,h} - 4A^{n-1}_{3,h} + A^{n-2}_{3,h}}{2\Delta t}, q_3\right) + d_3(\nabla A^n_{3,h}, \nabla q_3) - r_3\left(2A^{n-1}_{3,h} - A^{n-2}_{3,h}, q_3\right) - \Big(A^n_{3,h}(-a_{31}(2A^{n-1}_{3,h} \\ \quad -A^{n-2}_{3,h}) - a_{32}(2A^{n-1}_{1,h} - A^{n-2}_{1,h}) - a_{33}(2A^{n-1}_{2,h} - A^{n-2}_{2,h}), q_3)\Big) = 0, \quad \forall q_3 \in V_h. \end{cases} \tag{54}$$

In what follows, we shall report the numerical solution of the above scheme, which appears to be $\mathcal{O}(h^2 + \Delta t^2)$ using the $\mathbb{P}_2$ element.

6. Numerical verifications

In order to provide some verification of Theorem 2, we have performed calculations using our home-made code. Here, we have considered four model problems amenable to exact solutions in a unit square $\Omega = [0, 1]^2$ and a cubic box $\Omega = [-1, 1]^3$, that are given by

1. 2D

   (a)
$$A_1(x, y, t) = \cos(2\pi x)\cos(\pi y)\sin(2t),$$
$$A_2(x, y, t) = \cos(\pi x)\cos(2\pi y)\sin(2t),$$
$$A_3(x, y, t) = \cos(\pi x)\cos(\pi y)\sin(2t).$$

   (b)
$$A_1(x, y, t) = \cos(2\pi x)\cos(\pi y)\cos(t),$$
$$A_2(x, y, t) = \cos(\pi x)\cos(2\pi y)\cos(t),$$
$$A_3(x, y, t) = \cos(\pi x)\cos(\pi y)\cos(t).$$

2. 3D

   (a)
$$A_1(x, y, z, t) = \cos(2\pi x)\cos(\pi y)\cos(\pi z)\sin(2t),$$
$$A_2(x, y, z, t) = \cos(\pi x)\cos(2\pi y)\cos(\pi z)\sin(2t),$$
$$A_3(x, y, z, t) = \cos(\pi x)\cos(\pi y)\cos(2\pi z)\sin(2t).$$

   (b)
$$A_1(x, y, z, t) = \cos(2\pi x)\cos(\pi y)\cos(\pi z)\cos(t),$$
$$A_2(x, y, z, t) = \cos(\pi x)\cos(2\pi y)\cos(\pi z)\cos(t),$$
$$A_3(x, y, z, t) = \cos(\pi x)\cos(\pi y)\cos(2\pi z)\cos(t).$$

It is worth mentioning that the system (1) has no explicit closed form solutions because of the nonlinear reaction terms. In order to measure the error-norms, we first substituted the four above-mentioned examples into (1), and then obtained the time-dependent source terms. Notice that the initial data for Example 1(b) and Example 2(b) are non-zero values. The time t runs from [0, 1] with varying time step Δt. We set $d_i = r_i = a_{ij} = 1$, $1 \le i, j \le 3$. All the computational results are done on a uniform mesh layout.

In our numerical algorithm, we used the GMRES solver to deal with the system of equations (cf. Problem A and Problem B) resulting from the FE method. In conjunction with the SPARSKIT package taken from Saad [15],

Table 1
Summary of the 2D/3D meshes used in the calculation

Mesh types	Number of triangles		Number of nodes	Dimension of grided mesh $\mathbb{P}_2$ $\{A_i\}$
G_1	$\mathcal{T}_{\Delta,1}$	512	1 089	33×33
G_2	$\mathcal{T}_{\Delta,2}$	2 048	4 225	65×65
G_3	$\mathcal{T}_{\Delta,3}$	8 192	16 641	129×129
G_4	$\mathcal{T}_{\Delta,4}$	32 768	66 049	257×257
Mesh types	Number of tetrahedra		Number of nodes	Dimension of grided mesh $\mathbb{P}_2$ $\{A_i\}$
G_1	$\mathcal{T}_{\Delta,1}$	384	729	$9 \times 9 \times 9$
G_2	$\mathcal{T}_{\Delta,2}$	3 072	4 913	$17 \times 17 \times 17$
G_3	$\mathcal{T}_{\Delta,3}$	24 576	35 937	$33 \times 33 \times 33$
G_4	$\mathcal{T}_{\Delta,4}$	196 608	274 625	$65 \times 65 \times 65$

Table 2
Example 1(a): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\|\|A_1 - A_{1,h}\|\|_{\infty,0}$	Ratios	$\|\|A_1 - A_{1,h}\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.5393E−3	–	1.1937E−2	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	3.7810E−4	0.2456	2.9121E−3	0.2440
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	9.4016E−5	0.2487	7.2326E−4	0.2484
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	2.3471E−5	0.2496	1.8053E−4	0.2496
Mesh types	Mesh sizes	Time steps	$\|\|A_2 - A_{2,h}\|\|_{\infty,0}$	Ratios	$\|\|A_2 - A_{2,h}\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.5393E−3	–	1.1937E−2	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	3.7810E−4	0.2456	2.9121E−3	0.2440
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	9.4016E−5	0.2487	7.2326E−4	0.2484
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	2.3471E−5	0.2496	1.8053E−4	0.2496
Mesh types	Mesh sizes	Time steps	$\|\|A_3 - A_{3,h}\|\|_{\infty,0}$	Ratios	$\|\|A_3 - A_{3,h}\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.4930E−3	–	1.2285E−2	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	3.6917E−4	0.2473	3.0461E−3	0.2480
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	9.1900E−5	0.2489	7.5990E−4	0.2495
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	2.2949E−5	0.2497	1.8989E−4	0.2499

$h_0 = \max_{\widetilde{K} \in \mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a triangular element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the first-order scheme.

preconditioners for sparse GMRES iterative solvers derived from threshold-based ILUT factorizations were used. Unless otherwise specified, the following selective parameters were used for all performance calculations: for the G_1 and G_2 meshes, the number of fill-in elements per row was 17 and 33 respectively, while for the G_3 and G_4 meshes, the number was 50; calculation was terminated when the relative residual was below $\epsilon = 10^{-8}$; and convergence of the iterative process was fixed by an optimal number. The size of grid-points used in the calculations is summarized in Table 1.

To check the convergence rate with respect to the spatial discretization, we select the grid range from G_1 to G_4. The procedure was to simply reduce both the mesh size and the time step-size by half at each level of mesh refinement. From Tables 2–7, for different grid-spacings and time-step sizes, one could conclude the following:

- In Example 1(a) and (b), using the $\mathbb{P}_2$ element, the $|||\cdot|||_{\infty,0}$- and $|||\cdot|||_{0,1}$-errors of the three approximate species are $\mathcal{O}(h^2+\Delta t)$ using the first-order scheme, and when using the second-order scheme, the $|||\cdot|||_{\infty,0}$- and $|||\cdot|||_{0,1}$-errors of the three approximate species are $\mathcal{O}(h^{1.6}+\Delta t^{1.6})$ and $\mathcal{O}(h^{1.8}+\Delta t^{1.8})$, respectively.

Table 3
Example 1(b): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\Vert A_1 - A_{1,h} \Vert_{\infty,0}$	Ratios	$\Vert A_1 - A_{1,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	3.9777E−4	–	2.8878E−3	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	1.0535E−4	0.2649	6.3268E−4	0.2191
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.6697E−5	0.2534	1.5085E−4	0.2384
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	6.6968E−6	0.2508	3.7217E−5	0.2467
Mesh types	Mesh sizes	Time steps	$\Vert A_2 - A_{2,h} \Vert_{\infty,0}$	Ratios	$\Vert A_2 - A_{2,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	3.9777E−4	–	2.8878E−3	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	1.0535E−4	0.2649	6.3268E−4	0.2191
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.6697E−5	0.2534	1.5085E−4	0.2384
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	6.6962E−6	0.2508	3.7215E−5	0.2467
Mesh types	Mesh sizes	Time steps	$\Vert A_3 - A_{3,h} \Vert_{\infty,0}$	Ratios	$\Vert A_3 - A_{3,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	3.5139E−4	–	2.1665E−3	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	9.2817E−5	0.2641	5.7168E−4	0.2639
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.3504E−5	0.2532	1.4494E−4	0.2535
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	5.8943E−6	0.2508	3.6370E−5	0.2509

$h_0 = \max_{\widetilde{K}\in\mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a triangular element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the first-order scheme.

Table 4
Example 2(a): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\Vert A_1 - A_{1,h} \Vert_{\infty,0}$	Ratios	$\Vert A_1 - A_{1,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	1.4958E−1	–	1.7257E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.8866E−2	0.1930	4.7688E−1	0.2763
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.8698E−3	0.0994	8.2483E−2	0.1730
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.6760E−4	0.0584	1.0655E−2	0.1299
Mesh types	Mesh sizes	Time steps	$\Vert A_2 - A_{2,h} \Vert_{\infty,0}$	Ratios	$\Vert A_2 - A_{2,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	1.4611E−1	–	1.7345E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.9191E−2	0.1998	4.8141E−1	0.2776
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.8915E−3	0.0991	8.3263E−2	0.1730
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.6760E−4	0.0580	1.0655E−2	0.1280
Mesh types	Mesh sizes	Time steps	$\Vert A_3 - A_{3,h} \Vert_{\infty,0}$	Ratios	$\Vert A_3 - A_{3,h} \Vert_{0,1}$	Ratios
G_1	h_0	δt	1.4561E−1	–	1.7276E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.9026E−2	0.1993	4.8032E−1	0.2780
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.8871E−3	0.0995	8.3196E−2	0.1732
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.6760E−4	0.0581	1.0690E−2	0.1285

$h_0 = \max_{\widetilde{K}\in\mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a tetrahedral element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the first-order scheme.

- In Example 2(a) and (b), using the $\mathbb{P}_2$ element, the $\Vert\cdot\Vert_{\infty,0}$- and $\Vert\cdot\Vert_{0,1}$-errors of the three approximate species are $\mathcal{O}(h^2 + \Delta t)$ using the first-order scheme. Using the $\mathbb{P}_2$ element, in Example 2(b), the $\Vert\cdot\Vert_{\infty,0}$- and $\Vert\cdot\Vert_{0,1}$-errors of the three approximate species are $\mathcal{O}(h^2 + \Delta t^2)$ using the second-order scheme. These results imply that our 3D computer codes worked well.

Table 5
Example 2(b): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\|\|\|A_1 - A_{1,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_1 - A_{1,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.4422E−1	–	1.8439E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.8309E−2	0.1963	5.2094E−1	0.2825
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.7815E−3	0.0983	9.0705E−2	0.1741
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.3303E−4	0.0478	1.1714E−2	0.1291
Mesh types	Mesh sizes	Time steps	$\|\|\|A_2 - A_{2,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_2 - A_{2,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.4168E−1	–	1.8533E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.8644E−2	0.2022	5.2585E−1	0.2827
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.8072E−3	0.0980	9.1565E−2	0.1741
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.3303E−4	0.0474	1.1714E−2	0.1279
Mesh types	Mesh sizes	Time steps	$\|\|\|A_3 - A_{3,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_3 - A_{3,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.4129E−1	–	1.8459E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{4}\delta t$	2.8540E−2	0.2020	4.7425E−1	0.2569
G_3	$\frac{1}{4}h_0$	$\frac{1}{16}\delta t$	2.8030E−3	0.0982	9.1489E−2	0.1929
G_4	$\frac{1}{8}h_0$	$\frac{1}{64}\delta t$	1.3303E−4	0.0475	1.1714E−2	0.1280

$h_0 = \max_{\widetilde{K}\in\mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a tetrahedral element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the first-order scheme.

Table 6
Example 1(b): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\|\|\|A_1 - A_{1,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_1 - A_{1,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.9112E−4	–	2.0146E−3	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	8.3745E−5	0.4382	3.5166E−4	0.1746
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	3.5288E−5	0.4214	8.4384E−5	0.2400
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.3554E−5	0.3841	2.4344E−5	0.2885
Mesh types	Mesh sizes	Time steps	$\|\|\|A_2 - A_{2,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_2 - A_{2,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.9113E−4	–	2.0146E−3	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	8.3745E−5	0.4382	3.5166E−4	0.1746
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	3.5288E−5	0.4214	8.4384E−5	0.2400
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.3554E−5	0.3841	2.4343E−5	0.2884
Mesh types	Mesh sizes	Time steps	$\|\|\|A_3 - A_{3,h}\|\|\|_{\infty,0}$	Ratios	$\|\|\|A_3 - A_{3,h}\|\|\|_{0,1}$	Ratios
G_1	h_0	δt	1.9154E−4	–	9.0925E−4	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	8.9984E−5	0.4698	2.6717E−4	0.2938
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	3.8674E−5	0.4298	8.2442E−5	0.3086
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.4878E−5	0.3847	2.5267E−5	0.3069

$h_0 = \max_{\widetilde{K}\in\mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a triangular element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the second-order scheme.

7. Conclusion

The purpose of the study is to shed some light on the numerical schemes of the 3-species Lotka–Volterra competition diffusion model. We have shown that when the $\mathbb{P}_2$ element associated with the semi-implicit scheme is considered, $\mathcal{O}(h^2 + \Delta t)$ results with respect to the $\|\|\| \cdot \|\|\|_{\infty,0}$- and $\|\|\| \cdot \|\|\|_{0,1}$-error norms are attainable. We also extend the results to the second-order scheme. Using the $\mathbb{P}_2$ element, the numerical simulations suggest that $\mathcal{O}(h^2 + \Delta t^2)$ is achievable.

Table 7
Example 2(b): Convergence results for three competing species

Mesh types	Mesh sizes	Time steps	$\Vert A_1 - A_{1,h}\Vert_{\infty,0}$	Ratios	$\Vert A_1 - A_{1,h}\Vert_{0,1}$	Ratios
G_1	h_0	δt	0.1504E00	–	1.8523E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	2.8907E−2	0.1922	5.1709E−1	0.2792
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	2.8075E−3	0.0971	9.0142E−2	0.1743
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.2971E−4	0.0462	1.1667E−2	0.1294
Mesh types	Mesh sizes	Time steps	$\Vert A_2 - A_{2,h}\Vert_{\infty,0}$	Ratios	$\Vert A_2 - A_{2,h}\Vert_{0,1}$	Ratios
G_1	h_0	δt	0.1474E00	–	1.8612E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	2.9294E−2	0.1987	5.2202E−1	0.2804
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	2.8447E−3	0.0963	9.0997E−2	0.1743
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.2971E−4	0.0466	1.1667E−2	0.1282
Mesh types	Mesh sizes	Time steps	$\Vert A_3 - A_{3,h}\Vert_{\infty,0}$	Ratios	$\Vert A_3 - A_{3,h}\Vert_{0,1}$	Ratios
G_1	h_0	δt	0.1469E00	–	1.8538E00	–
G_2	$\frac{1}{2}h_0$	$\frac{1}{2}\delta t$	2.9195E−2	0.1987	5.2081E−1	0.2810
G_3	$\frac{1}{4}h_0$	$\frac{1}{4}\delta t$	2.8407E−3	0.0963	9.0922E−2	0.1746
G_4	$\frac{1}{8}h_0$	$\frac{1}{8}\delta t$	1.2971E−4	0.0467	1.1667E−2	0.1283

$h_0 = \max_{\widetilde{K}\in\mathcal{T}_h}\{h_{\widetilde{K}} : h_{\widetilde{K}} = \mathrm{diam}(\widetilde{K})\} \in G_1$, where $\widetilde{K}$ is a tetrahedral element, and $\delta t = 0.1$ using the $\mathbb{P}_2$ element with the second-order scheme.

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References

[1] J.G. Skellam, Random dispersal in theoretical populations, Biometrika 38 (1951) 196–218.
[2] S.A. Levin, Dispersion and population interactions, Amer. Natur. 108 (1974) 207–228.
[3] J.H. Steele, Spatial heterogeneity and population stability, Nature 248 (1974) 83.
[4] K. Kishimoto, The diffusive Lotka–Volterra system with three species can have a stable non-constant equilibrium solution, J. Math. Biol. 16 (1982) 103–112.
[5] Y. Kan-on, Existence and instability of Neumann layer solutions for a 3-component Lotka–Volterra model with diffusion, J. Math. Anal. Appl 243 (2000) 357–372.
[6] S. Martínez, The effect of diffusion for the multispecies Lotka–Volterra competition model, Nonlinear Anal. RWA 4 (2003) 409–436.
[7] S. Martínez, W.M. Ni, Periodic solutions for a 3×3 competitive system with cross-diffusion, Discrete Contin. Dyn. Syst. 15 (2006) 725–746.
[8] N.S. Goel, S.C. Maitra, E.W. Montroll, On the Volterra and other nonlinear models of interacting populations, Rev. Modern Phys. 43 (1971) 231–276.
[9] R.M. May, W.J. Leonard, Nonlinear aspects of competition between three species, SIAM J. Appl. Math. 29 (1975) 243–253.
[10] L. Frachebourg, P.L. Krapivsky, E. Ben-Naim, Spatial organization in cyclic Lotka–Volterra systems, Phys. Rev. E 54 (1996) 6186–6200.
[11] M. Gyllenberg, P. Yan, Y. Wang, A 3D competitive Lotka–Volterra system with three limit cycles: A falsification of a conjecture by Hofbauer and So, Appl. Math. Lett. 19 (2006) 1–7.
[12] M. Gyllenberg, P. Yan, Y. Wang, Limit cycles for competitor–competitor–mutualist Lotka–Volterra systems, Physica D 221 (2006) 135–145.
[13] B. Liu, On a finite element method for unsteady compressible viscous flows, Numer. Methods Partial Differential Equations 19 (2003) 152–166.
[14] B. Liu, On a finite element method for three-dimensional unsteady compressible viscous flows, Numer. Methods Partial Differential Equations 20 (2004) 432–449.
[15] Y. Saad, SPARSKIT: A basic tool-kit for sparse matrix computations (Version 2) http://www-users.cs.umn.edu/~saad/software/SPARSKIT/sparskit.html, 2005.